Some New Solutions of Shallow Water Wave Equation with Variable Coefficients By Generalized $\left(\frac{G'}{G}\right)$ -Expansion Method

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Abstract

In this work, the generalized $\binom{G'}{G}$ -expansion method is presented to seek some new exact solutions of Shallow Water wave equation with variable coefficients. As a result, more explicit traveling wave solutions involving arbitrary parameters are found out, which are expressed in terms of hyperbolic functions, the trigonometric functions and rational functions. When the parameters are taken special values, different types of wave solutions can be obtained.

Keywords: Variable Coefficients Shallow Water Wave equation, Generalized $\left(\frac{G'}{G}\right)$ -Expansion method, Exact Solutions.

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1 Introduction

The nonlinear partial differential equations in mathematical physics have played a major role in a great variety of contexts, such as physics, biology, engineering, fluid flow, signal processing, system identification, control theory, finance, and fractional dynamics. Wide classes of analytical methods have been proposed for finding the exact solutions of these nonlinear partial differential equations. For integrable non-linear differential equations, the inverse scattering transform method [1], the Hirota method [2], the truncated Painlevé expansion method [3], the Bäcklund transform method [4, 5] and the exp-function method [6, 7, 8] are used in looking for the exact solutions of nonlinear partial differential equations. Among non-integrable nonlinear differential equations, there is a wide class of the equations that referred to as the partially integrable, because these equations become integrable for some values of

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their parameters. There are many different methods to look for the exact solutions of these equations. The most famous algorithms are the Weierstrass elliptic function function method [9], the tanh- function method [10, 11] and the Jacobi elliptic function expansion method [12, 13] and so on.

However to our knowledge, most of aforementioned methods are related to the constant coefficient models. But the physical situations in which nonlinear equations arise tend to be highly idealized because of assumption of constant coefficients. Because of this, the study of nonlinear equations with variable coefficients has attracted much attention [14, 15, 16].

Among the other methods, a new method called Generalized $\left(\frac{G'}{G}\right)$ - Expansion method [17, 18] has been proposed to seek exact solutions of nonlinear partial differential equations. Being concise and straightforward, this method can be applied to various nonlinear partial differential equations with variable coefficients. As a result, hyperbolic function solutions, trigonometric function solutions and rational solutions with various parameters are obtained.

This method is based on the assumptions that the traveling wave solutions can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ where $G = G(\theta)$ satisfies the following second order linear ordinary differential equation

$$G''(\theta) + \lambda G'(\theta) + \mu G(\theta) = 0, \qquad (1.1)$$

where $\theta = p(t)x + q(t)$, p(t) and q(t) are functions to be determined.

This paper is organized as follows: In section 2, we summarize the generalized $\left(\frac{G'}{G}\right)$ expansion method. We apply the method to physically important nonlinear evolution
equation, namely, shallow water wave equation with variable coefficients in section
3 and abundant exact solutions are obtained which included the hyperbolic functions, the trigonometric functions and rational functions. Finally, we record some
concluding remarks in last section.

2 Description of The Generalized $\left(\frac{G'}{G}\right)$ -Expansion Method [17]

Consider the nonlinear partial differential equation with independent variables X = (x, y, z, ..., t) and dependent variable u in the following form:

$$F(u, u_t, u_x, u_y, u_z, \dots u_{xt}, u_{yt}, u_{zt}, u_{tt}, \dots) = 0,$$
(2.1)

where u = u(x, y, z, ...t) is an unknown function, F is a polynomial in u = u(x, y, z, ...t)and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of the generalized $\left(\frac{G'}{G}\right)$ -expansion method.

Step 1: We suppose that the solution of Eq. (2.1) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u = \alpha_0(X) + \sum_{i=1}^m \alpha_i(X) \left(\frac{G'}{G}\right)^i, \quad \alpha_m(X) \neq 0,$$
(2.2)

where $\alpha_0(X), \alpha_i(X), (i = 1, 2, ..., m)$ and $\theta = \theta(X)$ are all functions of X, to be determined later and $G = G(\theta)$ satisfies following equation

$$G''(\theta) + \lambda G'(\theta) + \mu G(\theta) = 0, \qquad (2.3)$$

where $\theta = p(t)x + q(t)$, p(t) and q(t) are functions to be determined.

Step 2: In order to determine u explicitly, we firstly find the value of integer m by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in Eq. (1.1).

Step 3: Substitute (2.2) along with Eq. (2.3) into Eq. (2.1) and collect all terms with the same order of $\binom{G'}{G}$ together, the left hand side of Eq. (2.1) is converted into a polynomial in $\binom{G'}{G}$. Then by setting each coefficient of this polynomial to zero, we derive a set of over-determined partial differential equations for $\alpha_0(X)$, $\alpha_i(X)$ and ζ . Step 4: Solve the system of over-determined partial differential equations obtained in Step 3 for $\alpha_0(X)$, $\alpha_i(X)$ and ζ .

Step 5: Use the results obtained in above steps to derive the solutions of Eq. (2.1) depending on $\left(\frac{G'}{G}\right)$, since the solutions of Eq. (2.3) have been well known to us depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, then the exact solutions of Eq. (2.1) are furnished.

3 Application of Generalized $\left(\frac{G'}{G}\right)$ -Expansion Method to Shallow Water Wave Equation with Variable Coefficients

In this section, we will use generalized $\left(\frac{G'}{G}\right)$ -expansion method to derive certain new solutions of shallow water wave equation with variable coefficients, which is very important nonlinear partial differential equations in the mathematical physics and have been paid attention by many researchers.

We start with the following shallow water wave equation with variable coefficients

$$u_t + \theta(t)u_x + \beta(t)uu_x + \sigma(t)u_xu_{xx} + \delta(t)u_{xxx} + \omega(t)uu_{xxx} + \rho(t)u_{xxxxx} = 0, \quad (3.1)$$

where $\theta(t), \beta(t), \sigma(t), \delta(t), \omega(t)$ and $\rho(t)$ are arbitrary functions of t. According to the method described above in section 2, by balancing the highest order nonlinear term and the highest order partial derivative term of u in Equation (3.1), we obtain m = 2. In order to search for explicit solutions, we suppose that equation (3.1) has the following solution expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u = \alpha_0(t) + \alpha_1(t) \left(\frac{G'}{G}\right) + \alpha_2(t) \left(\frac{G'}{G}\right)^2, \qquad (3.2)$$

where $\alpha_2(t) \neq 0$ and $G = G(\zeta)$. Substituting (3.2) into (3.1) and using (2.3), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq. (3.1) is converted into polynomial in $x^j \left(\frac{G'}{G}\right)$, (j = 0, 1). Setting each coefficient of

this polynomial to zero, we derive a set of overdetermined differential equations for $\alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), p(t)$ and q(t). Solving this set of equations, we have obtained the following results:

Case 1:

$$\begin{aligned} q(t) &= q(t), \quad \beta(t) = \beta(t), \quad \rho(t) = \rho(t), \quad \alpha_0(t) = c_3, \quad \alpha_1(t) = c_2\lambda, \quad \alpha_2(t) = c_2, \\ \omega(t) &= \frac{48\rho(t)c_1^2}{c_2}, \quad \omega(t) = \frac{-156\rho(t)c_1^2}{c_2}, \\ \delta(t) &= \frac{1}{12} \frac{48\rho(t)c_1^4c_2\lambda^2 + 384\rho(t)c_1^4c_2\mu - \beta(t)c_2^2 - 576\rho(t)c_1^4c_3}{c_1^2c_2}, \\ \theta(t) &= \frac{-(12q'(t) - 480\rho(t)c_1^5\lambda^2\mu + 12c_1\beta(t)c_3 - c_1c_2\lambda^2\beta(t) + 960c_1^5\mu^2\rho(t) + 60\rho(t)c_1^5\lambda^4 - 8c_1c_2\mu\beta(t))}{12c_1}, \end{aligned}$$

$$(3.3)$$

where c_1, c_2 and c_3 are arbitrary constants.

Substituting the general solutions of (2.3) into (3.2) and using (3.3), we have three types of exact solutions of (3.1) as follows:

When $\lambda^2 - 4\mu > 0$, we have obtained hyperbolic function solution in the form

$$u(x,t) = c_3 + c_2 \lambda \left(\frac{1}{2} \frac{\sqrt{(\lambda^2 - 4\mu)}(a_1 \sinh(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})) + a_2 \cosh(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})))}{(a_2 \sinh(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})) + a_1 \cosh(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})))} - \frac{\lambda}{2} \right) + c_2 \left(\frac{1}{2} \frac{\sqrt{(\lambda^2 - 4\mu)}(a_1 \sinh(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})) + a_2 \cosh(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})))}{(a_2 \sinh(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})) + a_1 \cosh(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})))} - \frac{\lambda}{2} \right)^2.$$

$$(3.4)$$



Figure 1: Graphical representation of solution (3.4), for $\lambda = 4, \mu = 2, c_2 = 2, c_3 = 1, a_1 = 1, a_2 = 2, c_1 = 2$ and q(t) = t

When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$u(x,t) = c_3 + c_2 \lambda \left(\frac{1}{2} \frac{\sqrt{(-\lambda^2 + 4\mu)}(-a_2 \sin(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)})) + a_1 \cos(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)})))}{(a_1 \sin(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)})) + a_2 \cos(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)})))} - \frac{\lambda}{2} \right) + c_2 \left(\frac{1}{2} \frac{\sqrt{(-\lambda^2 + 4\mu)}(-a_2 \sin(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)})) + a_1 \cos(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)})))}{(a_1 \sin(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)})) + a_2 \cos(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)})))} - \frac{\lambda}{2} \right)^2.$$

$$(3.5)$$

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Figure 2: Graphical representation of solution (3.5), when $\lambda = 2, \mu = 4, c_2 = 2, c_3 = 1, a_1 = 1, a_2 = 2, c_1 = 2$ and q(t) = t

When $\lambda^2 - 4\mu = 0$, we get rational solution as follows:

$$u(x,t) = c_3 + c_2 \lambda \left(\frac{A_2}{A_1 + A_2((c_1 x + q(t)))} - \frac{\lambda}{2} \right) + c_2 \left(\frac{A_2}{A_1 + A_2((c_1 x + q(t)))} - \frac{\lambda}{2} \right)^2,$$
(3.6)

where A_1, A_2, a_1 and a_2 are arbitrary constants.



Figure 3: Graphical representation of solution (3.6), when $\lambda = 2, c_2 = 2, c_3 = 1, A_1 = 1, A_2 = 2, c_1 = 2$ and q(t) = t

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Case 2:

$$q(t) = q(t), \quad \rho(t) = \rho(t), \quad \alpha_0(t) = c_3, \quad \alpha_1(t) = c_2(\lambda \pm 3\sqrt{-\lambda^2 + 4\mu}), \quad \alpha_2(t) = c_2, \\ \omega(t) = \frac{48\rho(t)c_1^2}{c_2}, \quad \sigma(t) = \frac{-156\rho(t)c_1^2}{c_2}, \\ \delta(t) = \frac{-(25\rho(t)c_1^2c_2\lambda^2 - 52\rho(t)c_1^2c_2\mu - 24\rho(t)c_1^2\lambda c_2(\lambda \pm 3\sqrt{-\lambda^2 + 4\mu}) + 48c_1^2c_3\rho(t))}{(c_1^2c_2\mu^2)^2 c_2^2 - 52\rho(t)c_1^2c_2\mu^2 - 24\rho(t)c_1^2c_2^2 - 52\rho(t)c_1^2c_2\mu^2)}, \\ \theta(t) = \frac{-(192\rho(t)\mu c_1^5c_3 + c_2q'(t) + 144\rho(t)\lambda^2\mu c_1^5c_2 - 192\rho(t)\mu^2 c_1^5c_2 - 24\rho(t)\lambda^4 c_1^5c_2 - 48\rho(t)\lambda^2 c_1^5c_3)}{c_1c_2} + \frac{-(24\rho(t)\lambda^3 c_1^5c_2(\lambda \pm 3\sqrt{-\lambda^2 + 4\mu}) - 96\rho(t)c_1^5c_2\mu\lambda(\lambda \pm 3\sqrt{-\lambda^2 + 4\mu}))}{c_1c_2},$$

$$(3.7)$$

where c_1, c_2 and c_3 are arbitrary constants.

Consequently, we have the following three types of exact solutions of (3.1): If $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution in the form

$$u(x,t) = c_3 + \left(\lambda \pm 3\sqrt{-\lambda^2 + 4\mu}\right)c_2 \left(\frac{1}{2}\frac{\sqrt{(\lambda^2 - 4\mu)}(a_1\sinh(\Lambda_1) + a_2\cosh(\Lambda_1))}{(a_2\sinh(\Lambda_1) + a_1\cosh(\Lambda_1))} - \frac{\lambda}{2}\right) + c_2 \left(\frac{1}{2}\frac{\sqrt{(\lambda^2 - 4\mu)}(a_1\sinh(\Lambda_1) + a_2\cosh(\Lambda_1))}{(a_2\sinh(\Lambda_1) + a_1\cosh(\Lambda_1))} - \frac{\lambda}{2}\right)^2.$$
(3.8)

If $\lambda^2 - 4\mu < 0$, we have trigonometric function solution in the following form:

$$u(x,t) = c_3 + (\lambda \pm 3\sqrt{-\lambda^2 + 4\mu})c_2 \left(\frac{1}{2} \frac{\sqrt{(-\lambda^2 + 4\mu)}(-a_2 \sin(\Lambda_2) + a_1 \cos(\Lambda_2))}{(a_1 \sin(\Lambda_2) + a_2 \cos(\Lambda_2))} - \frac{\lambda}{2}\right) + c_2 \left(\frac{1}{2} \frac{\sqrt{(-\lambda^2 + 4\mu)}(-a_2 \sin(\Lambda_2) + a_1 \cos(\Lambda_2))}{(\Lambda_2) + a_2 \cos(\Lambda_2))} - \frac{\lambda}{2}\right)^2.$$
(3.9)

If $\lambda^2 - 4\mu = 0$, we get rational solution as follows:

$$u(x,t) = c_3 + c_2(\lambda \pm 3\sqrt{-\lambda^2 + 4\mu}) \left(\frac{A_2}{A_1 + A_2((c_1x + q(t)))} - \frac{\lambda}{2}\right) + c_2\left(\frac{A_2}{A_1 + A_2((c_1x + q(t)))} - \frac{\lambda}{2}\right)^2$$
(3.10)
where A_1, A_2, a_1, a_2 are arbitrary constants and $\Lambda_1 = \frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2 - 4\mu)})$ and $\Lambda_2 = \frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)}).$

Case 3:

$$q(t) = q(t), \quad \rho(t) = \rho(t), \quad \alpha_0(t) = c_3, \quad \alpha_1(t) = c_2(\lambda \pm \sqrt{(\lambda^2 - 4\mu)}), \quad \alpha_2(t) = c_2, \\ \omega(t) = \frac{48\rho(t)c_1^2}{c_2}, \quad \sigma(t) = \frac{-156\rho(t)c_1^2}{c_2}, \\ \delta(t) = \frac{-(-24\rho(t)c_1^2c_2\lambda((\lambda \pm \sqrt{(\lambda^2 - 4\mu)})) + 13\rho(t)c_1^2c_2\lambda^2 - 4\rho(t)c_1^2\mu c_2 + 48c_1^2c_3\rho(t))}{c_2}, \\ \theta(t) = \frac{-(-36\rho(t)\lambda^4c_2c_1^5 + c_2q'(t) + 216\rho(t)c_1^5c_2\lambda^2\mu - 288\rho(t)c_2\mu^2c_1^5 - 144\rho(t)\lambda\mu c_1^5(c_2(\lambda \pm \sqrt{(\lambda^2 - 4\mu)})))}{c_1c_2} + \frac{-(288\rho(t)\mu c_1^5c_3 - 72\rho(t)\lambda^2c_1^5c_3 + 36\rho(t)\lambda^3c_1^5c_2(\lambda \pm \sqrt{(\lambda^2 - 4\mu)}))}{c_1c_2}, \end{cases}$$

$$(3.11)$$

where c_1, c_2 and c_3 are arbitrary constants. When $\lambda^2 - 4\mu > 0$, we derived hyperbolic function solution of Eq. (3.1) in the

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following form

$$u(x,t) = c_3 + c_2(\lambda \pm \sqrt{(\lambda^2 - 4\mu)}) \left(\frac{1}{2} \frac{\sqrt{(\lambda^2 - 4\mu)}(a_1 \sinh(\Lambda_1) + a_2 \cosh(\Lambda_1))}{(a_2 \sinh(\Lambda_1) + a_1 \cosh(\Lambda_1))} - \frac{\lambda}{2} \right) + c_2 \left(\frac{1}{2} \frac{\sqrt{(\lambda^2 - 4\mu)}(a_1 \sinh(\Lambda_1) + a_2 \cosh(\Lambda_1))}{(a_2 \sinh(\Lambda_1) + a_1 \cosh(\Lambda_1))} - \frac{\lambda}{2} \right)^2.$$
(3.12)

When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution of Eq. (3.1) as follows:

$$u(x,t) = c_3 + c_2(\lambda \pm \sqrt{(\lambda^2 - 4\mu)}) \left(\frac{1}{2} \frac{\sqrt{(-\lambda^2 + 4\mu)}(-a_2 \sin(\Lambda_2) + a_1 \cos(\Lambda_2))}{(a_1 \sin(\Lambda_2) + a_2 \cos(\Lambda_2))} - \frac{\lambda}{2}\right) + c_2 \left(\frac{1}{2} \frac{\sqrt{(-\lambda^2 + 4\mu)}(-a_2 \sin(\Lambda_2) + a_1 \cos(\Lambda_2))}{(a_1 \sin(\Lambda_2) + a_2 \cos(\Lambda_2))} - \frac{\lambda}{2}\right)^2.$$
(3.13)

When $\lambda^2 - 4\mu = 0$, we get rational solution of Eq. (3.1) as follows:

$$u(x,t) = c_3 + c_2(\lambda \pm \sqrt{(\lambda^2 - 4\mu)}) \left(\frac{A_2}{A_1 + A_2((c_1x + q(t)))} - \frac{\lambda}{2}\right) + c_2 \left(\frac{A_2}{A_1 + A_2((c_1x + q(t)))} - \frac{\lambda}{2}\right)^2$$
(3.14)

where A_1, A_2, c_1, c_2 and c_3 are arbitrary constants $\Lambda_1 = \frac{1}{2}((c_1x + q(t))\sqrt{\lambda^2 - 4\mu})$ and $\Lambda_2 = \frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2 + 4\mu)}).$

4 Conclusion and Remarks

In this article, we have established certain new solutions of the Shallow water wave equation (3.1) by using generalized $\left(\frac{G'}{G}\right)$ -expansion method. These solutions are expressed in terms of hyperbolic, trigonometric, and rational functions involving arbitrary parameters. Also, we have presented graphical representation of solutions (3.4), (3.5) and (3.6) in Figure 1, Figure 2, Figure 3 respectively. In the similar way we can represent graphically the behavior of other derived solutions. It is imperative to state that some of our solutions are expressed in terms of physical parameter q(t) and $\rho(t) = t$ and it plays a crucial role as the remaining coefficients of equation (3.1) have all been expressed in terms of it. In fact, the various other arbitrary constants occurring in the solutions, along with q(t), $\rho(t)$, provide further freedom to simulate the desired physical situations. In almost all the cases the solutions obtained may also help to recover certain solutions available in literature for the particular models with constant coefficients. It is worth to mention here that the correctness of the solutions has been checked with the aid of software MAPLE.

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